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On Approximation by Discrete Semigroups

NAZAR H. ABDELAZIZ

Department of Mathematics, University of California at Berkeley, Berkeley, California 97420, U.S.A.

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The present paper deals with the problem of approximation of a continuous parameter semigroup $T(t)$, $t > 0$ on a Banach space X by means of a sequence of discrete parameter semigroups (F_n^k), where F_n is a bounded operator on a Banach space X_n, $n \in N$, and where (X_n) and X are related in some appropriate sense. This problem arises, e.g., when numerical methods are used to approximate solutions of initial boundary value problems in PDEs. The results obtained here present a new set of tests for convergence of discrete semigroups, which are different from those in (E. Görlich and D. Pontzen, *Tôhuku Math. J. (2)* **34**, No. 4 (1982), 539–552). Theorem 2 and its corollaries extend the earlier results on this point.

INTRODUCTION

Let $(X, \|\cdot\|)$ be a Banach space, and $(X_n, \|\cdot\|_n)$ a sequence of Banach spaces approximating X in the following sense: There exist bounded linear operators $P_n : X \to X_n$, $n \in N$ such that for each $x \in X$

$$\lim_{n \to \infty} \|P_n x\|_n = \|x\|.$$

In particular there is a constant $\beta > 0$ such that

$$\|P_n x\|_n \leqslant \beta \, \|x\|, \qquad \forall n \in N, \qquad x \in X.$$

Further, let (ρ_n) be a sequence of positive numbers tending to 0, and for each $n \in N$, let $F_n : X_n \to X_n$ be a bounded linear operator on X_n. We consider conditions under which the sequence of discrete parameter semigroups (F_n^k) may converge in some sense to a continuous parameter semigroup $T(t)$, $t > 0$ defined on X. This problem arises, for example, in

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applications where numerical methods are used to approximate solutions of linear initial boundary value problems by means of systems of finite difference equations. One of the basic assumptions in the earlier investigations of this problem (cf. [3, 7, 10, 11, 13, 16]) was derived from the stability condition of such systems, and may be formulated in functional analytic language (Trotter [16]) as

$$\|F_n^k\|_n \leqslant M e^{\omega k \rho_n}, \qquad n, k \in N, \tag{1}$$

where M and ω are independent of n and k. A relatively recent study by Görlich and Pontzen [6] shows that convergence of a sequence of discrete semigroups may still hold under weaker stability conditions than that in (1).

The object of the present paper is to study this problem using stability conditions different from those in [6], but which are weaker than (1). Meawhile, a generalized form of limit inferior of sequences of operators is employed, so that one obtains a more general theory than that which is based on the existence of limits of such sequences. We remark that in this case the limit semigroup $T(t)$ is not in general C_0, but rather of class (1, A). We may also add that Theorems 1–3 and the corollaries below form a new set of tests for convergence when it comes to the use of numerical methods in solving initial boundary value problems. A strong version of Trotter's Theorem on approximation of C_0 semigroups by means of discrete semigroups [16] is also obtained in Corollary 2 of Theorem 2.

Section 1 is a brief review of the necessary notations and definitions. Section 2 deals with a question concerning the boundedness of Riemann sums. This will be useful in proving the convergence theorems of Section 3.

An example in discussed at the end. It arises from the problem of discretization of a certain Cauchy problem (cf. Sunouchi [15]).

1. Preliminaries

By a limit of a sequence of vectors (x_n), $x_n \in X_n$, we mean an element $x \in X$ defined as

$$\widetilde{\lim}\, x_n = x \Leftrightarrow \lim_{n \to \infty} \|P_n x - x_n\|_n = 0.$$

Consider a sequence of operators (A_n), $A_n: X_n \to X_n$. The limit of the sequence (A_n), denoted by $A = \lim A_n$ (cf. [16]), is an operator whose domain consists of all $x \in X$ for which there is an element $y \in X$, such that $P_n x \in D(A_n)$ and $\widetilde{\lim}\, A_n P_n x = y$, where by definition $Ax = y$. A more general procedure of forming limits of sequences of operators is that of limit inferior, denoted by $\hat{A} = \liminf A_n$. This is an operator (possibly multi-valued) from X to itself that is defined as follows (see also [1] and [11]): For any elements $x, y \in X$, $y \in (\liminf A_n)\, x$ iff there exists a sequence (x_n), $x_n \in X_n$, such that $\widetilde{\lim}\, x_n = x$ and $\widetilde{\lim}\, A_n x_n = y$. As an operator, the limit inferior is an extension of the limit of the sequence (A_n). We shall also make use of the following set (cf. [1]):

$$\begin{aligned} D^\circ = \{x \in X\colon & \text{there exists a sequence } (x_n), \\ & x_n \in D(A_n) \text{ such that } \widetilde{\lim}\, x_n = x \\ & \text{and } \sup_n \|A_n x_n\|_n < \infty\}. \end{aligned}$$

Recall that a semigroup of linear operators on a Banach space X is a mapping $T(t)\colon (0, \infty) \to L(X)$, (where $L(X)$ is the space of bounded linear operators on X), satisfying $T(t+s) = T(t)\, T(s)$, for all $t, s > 0$. We assume in this discussion that the semigroup is a strongly continuous map on $(0, \infty)$. The infinitesimal operator of $T(t)$ is defined as usual by

$$A_o x = \lim_{x \to 0+} h^{-1}(T(h) - I)\, x$$

whenever the limit exists. The closure $\bar{A}_o$, when it exists, is called the infinitesimal generator (i.g.) of $T(t)$. The type of $T(t)$, denoted ω_0 is given by

$$\omega_0 = \liminf_{t > 0} t^{-1} ln \|T(t)\|.$$

A C_0 semigroup is one that is strongly continuous at $t = 0$, where by definition $T(0) = I$. A semigroup $T(t)$, $t > 0$ is of class $(1, \mathrm{A})$ if it satisfies the following:

$$\int_0^1 \|T(t)\|\, dt < \infty, \qquad \lim_{\lambda \to \infty} \lambda \int_0^\infty e^{-\lambda t} T(t)\, x\, dt = x, \qquad x \in X.$$

Every C_0 semigroup is of class $(1, \mathrm{A})$, but the converse is not true in general (cf. [14]). For further details and information on this subject we refer to [3, 5, 8]. For convenience we quote here Theorem 2 of [1].

THEOREM A (cf. [1, Theorem 2]). *Let $T_n(t)$, $t > 0$ be a semigroup of class $(1, \mathrm{A})$ that is defined on X_n and with* i.g. *A_n, such that the following conditions are satisfied: There exist constants M, $C > 0$ and $\omega \geq 0$ such that*

$$\int_0^\infty e^{-\omega t} \|T_n(t)\|_n\, dt \leq M, \qquad n \in N, \tag{$*$}$$

and

$$\|\lambda R(\lambda; A_n)\|_n \leqslant C, \qquad \lambda \geqslant \omega, n \in N. \tag{**}$$

Then the following assertions are equivalent:

(i) *There exists a semigroup* $T(t)$ *of class* (1, A) *defined on* X, *such that for each* x *in* X *and* x_n *in* X_n, $n = 1, 2, \ldots$

$$\widetilde{\lim}\, x_n = x \Rightarrow \widetilde{\lim}\, T_n(t)\, x_n = T(t)\, x, \qquad t > 0$$

uniformly on compact subsets of $(0, \infty)$,

(ii) D^c *and* $R(\lambda_0 I - \hat{A})$ *are dense in* X *for some* $\lambda_0 > \omega$.

In either case $\hat{A}$ *is the* i.g. *of* $T(t)$.

Remark. In proving (ii) $\Rightarrow$ (i), it was merely shown in [1] that the limit given by (i) holds for $z \in D(\hat{A}^2)$ and (z_n) satisfying Proposition 1-d of [1]. Since the main results in the present paper depend to a large extent on this formula, we find it convenient to supply here some details concerning this proof and which were not given in [1].

Let us recall Eqs. (11) and (12) of [1], which define the semigroups $T(t)$ and $T_n(t)$ in terms of their respective resolvent operators; namely, that for some $\gamma > \omega$

$$T(t)\, z = z + t\hat{A}z + \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} R(\lambda; \hat{A})\, \hat{A}^2 z \frac{d\lambda}{\lambda^2}, \qquad z \in D(\hat{A}^2), \tag{11A}$$

$$T_n(t)\, z_n = z_n + tA_n z_n + \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma - i\infty} e^{\lambda t} R(\lambda; A_n) \times A_n^2 z_n \frac{d\lambda}{\lambda^2}, \qquad z_n \in D(A_n^2), \tag{12A}$$

where z and (z_n) are as in Proposition 1-d of [1]. We note from the proof of Theorem 2 in [1] that the integrand in (12A) is dominated in norm by $\alpha M e^{\gamma t}/|\lambda|^2$, which is bounded on a given interval $[a, b]$ by $\alpha M e^{\gamma b}/|\lambda|^2$. It follows from this, (11A), and (12A) that $\widetilde{\lim}\, T_n(t)\, z_n = T(t)\, z$ holds uniformly on compacts.

Now, let $x \in X$ and $x_n \in X_n$ be such that $\widetilde{\lim}\, x_n = x$. Since $D(\hat{A}^2)$ is dense in X (note that $\lambda^2 R^2(\lambda; \hat{A})\, x \to x$, $\forall x \in X$, cf. [1]), we find that: Given $\varepsilon > 0$ there is a $z \in D(\hat{A}^2)$ such that $\|x - z\| < \varepsilon$. By the above argument, there is a sequence (z_n), $z_n \in D(A_n^2)$, such that $\widetilde{\lim}\, z_n = z$ and $\widetilde{\lim}\, T_n(t)\, z_n = T(t)\, z$ uniformly on compact subsets of $(0, \infty)$. Moreover, by Theorem 7.4.4 of [8] (see also [12, Theorem 4.3]), we have the following: If $[a, b] \subset (0, \infty)$ then

$$\|T_n(t)\|_n \leqslant C_1 M^2 e^{\omega b}/t^2, \qquad \forall t \in [a, b], \qquad n \in N.$$

Therefore, the desired result is obtained from the following inequality:

$$\|T_n(t)\,x_n - P_n T(t)\,x\|_n \leqslant \|T_n(t)(x_n - z_n)\|_n + \|T_n(t)\,z_n - P_n T(t)\,z\|_n + \|P_n T(t)(z-x)\|_n.$$

It is worth noting here, that conditions $(*)$ and $(**)$ were first used by I. Miyadera [12].

2. Boundedness of Riemann Sums

We consider here the problem of uniform boundedness of Riemann sums of the type $\sum_{j=1}^{\infty} \rho_n \psi(j\rho_n)$, where (ρ_n) is a null sequence of positive numbers and $\psi(t)$ is a non-negative function that is defined on $(0, \infty)$. The first result deals with the case of monotone functions and the proof is rather simple. We state it as a lemma.

LEMMA 1. *Let $\psi(t) \in L^1(0, \infty)$ be a non-negative non-increasing function, and let (ρ_n) be any null sequence of positive numbers. Then the sums $\sum_{j=1}^{\infty} \rho_n \psi(j\rho_n)$ are bounded uniformly in n.*

Proof. $\sum_{j=1}^{\infty} \rho_n \psi(j\rho_n) \leqslant \sum_{j=0}^{\infty} \int_{j\rho_n}^{(j+1)\rho_n} \psi(t)\,dt = \int_0^{\infty} \psi(t)\,dt.$ ∎

Another result in this direction was given by P. Chernoff in a private communication with the author. We summarize that result in the remainder of the section.

The following result, which is a corollary of "The Martingale Convergence Theorem," was obtained by B. Jessen [9].

THEOREM B. (B. Jessen). *Let f be periodic with period* 1 *and integrable in the Lebesgue sense over* (0, 1). *Then, with $S_n(f; x) = (1/n) \sum_{j=1}^{n} f(x + j/n)$, the sequence $S_{2^n}(f; x)$ converges to $\int_0^1 f(x)\,dx$ as $n \to \infty$ for a.e. x.*

COROLLARY 1. (P. Chernoff). *Let $0 \leqslant f \in L^1(-\infty, \infty)$ then for a.e. $x \in \mathfrak{R}$,*

$$\frac{1}{2^n} \sum_{-\infty}^{\infty} f\left(x + \frac{j}{2^n}\right) \to \int_{-\infty}^{\infty} f(x)\,dx.$$

Proof. Define $F(x) := \sum_{m=-\infty}^{\infty} f(x+m)$. Note that $F(x+1) = F(x)$. Also,

$$\int_0^1 F(x)\,dx = \sum_{-\infty}^{\infty} \int_0^1 f(x+m)\,dx = \int_{-\infty}^{\infty} f(x)\,dx < \infty.$$

So $F \in L^1(0, 1)$. Therefore, by Jessen's Theorem, for a.e. x,

$$\frac{1}{2^n} \sum_{j=1}^{2^n} F\left(x + \frac{j}{2^n}\right) \to \int_0^1 F(x+s)\,ds = \int_{-\infty}^{\infty} f(s)\,ds,$$

but the left side is precisely $(1/2^n) \sum_{j=-\infty}^{\infty} f(x + j/2^n)$. ∎

Now, suppose that $f \in L^1(0, \infty)$. Extend f to be $=0$ for $x<0$, then from Corollary 1 applied to f we have:

COROLLARY 2. (P. Chernoff). *If* $f \in L^1(0, \infty)$, *then for* a.e. $x \in \mathfrak{R}$,

$$\frac{1}{2^n} \sum_{x+j/2^n > 0} f(x + j/2^n) \to \int_0^{\infty} f(x)\,dx.$$

3. CONVERGENCE OF DISCRETE SEMIGROUPS

In what follows, F_n denotes a bounded operator on the Banach space X_n, A_n the operator defined by $A_n := \rho_n^{-1}(F_n - I)$, $\rho_n > 0$, and $\hat{A} :=$ lim inf A_n.
To furter simplify the notation we write

$$Q_{nj}(t) = e^{-t/\rho_n} \left(\frac{t}{\rho_n}\right)^j \frac{1}{j!},$$

where t, $\rho_n > 0$, $n = 1, 2, ...$ and j is a non-negative integer. The main results here are Theorems 1–3. Theorem 2 extends (and its proof depends on) Theorem 1. Theorem 3 is rather independent. Some important consequences appear in the corollaries. First, we state a lemma. The proof is elementary and we omit it.

LEMMA 2. *Let k be a positive integer, then*

$$\sum_{j=0}^{\infty} \frac{k^j}{j!} (j-k)^4 = e^k(3k^2 + k),$$

and

$$\sum_{j=0}^{\infty} \frac{k^j}{j!} (j-k)^2 = ke^k.$$

THEOREM 1. *Let $0 \leqslant \psi(t)$, $t \in (0, \infty)$ be a non-increasing function, such that $\psi(t) \in L^1(0, \infty) \cap L^p(0, \infty)$ for some $p > 1$. Further, let (F_n), $F_n \in L(X_n)$, $n = 1, 2, ...$ be a sequence of operators and (ρ_n) a null sequence of positive numbers, such that following conditions are satisfied:*

(i) $\|F_n^k\|_n \leqslant \psi(k\rho_n)$, $\forall k, n \in N$,

(ii) *There exist constants* $C > 0$ *and* $\omega \geqslant 0$ *such that*

$$\|\lambda R(\lambda; A_n)\|_n \leqslant C, \qquad n \in N, \quad \lambda \geqslant \omega,$$

(iii) D° *and* $R(\lambda_0 I - \hat{A})$ *are dense in* X, *for some* $\lambda_0 > \omega$.

Then $\hat{A}$ *is the* i.g. *of a semigroup* $T(t)$ *of class* (1, A) *defined on* X. *Moreover, for each* $x \in X$ *and* $x_n \in X_n$, $n \in N$:

$$\widetilde{\lim}\, x_n = x \Rightarrow \widetilde{\lim}\, F_n^{[t/\rho_n]} x_n = T(t)\, x. \tag{2}$$

Proof. Let $T_n(t) = e^{tA_n}$ denote the semigroup on X_n that is generated by A_n. We begin by showing that $\hat{A}$ is the i.g. of a semigroup $T(t)$ of class (1, A), (a fortiori $\hat{A}$ is single valued) such that

$$\widetilde{\lim}\, x_n = x \Rightarrow \widetilde{\lim}\, T_n(t)\, x_n = T(t)\, x, \qquad t > 0 \tag{3}$$

uniformly on compact t-intervals. For this, it suffices to show that condition (∗) of Theorem A holds, since (∗∗) is already assumed in (ii). Now,

$$\|T_n(t)\|_n \leqslant \sum_{k=0}^{\infty} Q_{nk}(t)\, \|F_n^k\|_n \leqslant e^{-t/\rho_n} + \sum_{k=1}^{\infty} Q_{nk}(t)\, \psi(\rho_n k).$$

A routine calculation along with Lemma 1 then shows that

$$\begin{aligned}\int_0^\infty e^{-\omega t}\, \|T_n(t)\|_n\, dt &\leqslant \int_0^\infty \|T_n(t)\|_n\, dt \leqslant \rho_n + \sum_{k=1}^{\infty} \rho_n \psi(\rho_n k)\\ &\leqslant M_1 + \int_0^\infty \psi(t)\, dt,\end{aligned}$$

where M_1 is some constant.

To prove (2), we show first that if $x' \in D^\circ$, then there is a sequence (x'_n), $x'_n \in X_n$ such that $\widetilde{\lim}\, x'_n = x'$, and

$$\lim_{n \to \infty} \|\exp(\rho_n k_n A_n) - F_n^{k_n})\, x'_n\|_n = 0, \tag{4}$$

where for a given $t > 0$, $k_n = [t/\rho_n]$, so that $\rho_n k_n \leqslant t$, $\forall n$ and $\rho_n k_n \to t$ as $n \to \infty$.

Let $x' \in D^\circ$. Then, by definition, there is a sequence (x'_n) such that $\widetilde{\lim}\, x'_n = x'$ and $\sup_n \|A_n x'_n\|_n < \infty$. Note that $\|x'_n\|_n \leqslant \beta_1 \|x'\|$, $n \in N$ for some constant β_1 (depending on x as well as on the sequence (x'_n)). We have the following:

$$\begin{aligned}\|(\exp(\rho_n k_n A_n) - F_n^{k_n})\, x_n'\|_n &\leqslant \sum_{j=0}^{\infty} Q_{nj}(\rho_n k_n)\, \|(F_n^j - F_n^{k_n})\, x_n'\|_n \\ &\leqslant \sum_{j=1}^{\infty} Q_{nj}(\rho_n k_n)\, \|(F_n^j - F_n^{k_n})\, x_n'\|_n \\ &\quad + e^{-k_n}\, \|(I - F_n^{k_n})\, x_n'\|_n \\ &= I_1 + I_2.\end{aligned}$$

Here, we may consider only the non-zero values of k_n, since this can always be achieved by taking n sufficiently large. Also, note that the left side is trivially equal to 0 when $k_n = 0$. Now,

$$\begin{aligned} I_2 &= e^{-k_n} \left\| \sum_{m=1}^{k_n} (F_n^{m-1} - F_n^m)\, x_n' \right\|_n \leqslant e^{-k_n} \rho_n\, \|A_n x_n'\|_n \sum_{m=1}^{k_n} \|F_n^{m-1}\|_n \\ &\leqslant e^{-k_n} \{\sup_n \|A_n x_n'\|_n\} \left\{ \rho_n + \sum_{m=1}^{k_n - 1} \rho_n \psi(m\rho_n) \right\} \\ &\leqslant e^{-k_n} \{\sup_n \|A_n x_n'\|_n\} \left\{ \rho_n + \int_0^{\infty} \psi(t)\, dt \right\}.\end{aligned}$$

Hence, $I_2 \to 0$ as $n \to \infty$. For I_1, we follows an argument that was used in [2, Lemma 2.5]. We have for $\varepsilon \in (0, 1)$

$$\begin{aligned} I_1 &= \sum_{j=1}^{\infty} Q_{nj}(\rho_n k_n) \cdot \|(F_n^j - F_n^{k_n})\, x_n'\|_n \\ &= \left(\sum_1 + \sum_2 \right) Q_{nj}(\rho_n k_n) \cdot \|(F_n^j - F_n^{k_n})\, x_n'\|_n,\end{aligned}$$

where

$$\sum_1 := \sum_{|j - k_n| > \varepsilon k_n}, \qquad \sum_2 := \sum_{|j - k_n| \leqslant \varepsilon k_n},$$

and where it is noted in view of [3, p. 18] that

$$\sum_{|j-u| > \delta} \frac{u^j}{j!} \leqslant u e^u / \delta^2.$$

Therefore, we have

$$\begin{aligned} \sum_1 Q_{nj}(\rho_n k_n)\, \|(F_n^j - F_n^{k_n})\, x_n'\|_n &\leqslant \|x_n'\|_n \sum_1 Q_{nj}(\rho_n k_n)(\|F_n^j\|_n + \|F_n^{k_n}\|_n) \\ &\leqslant \beta_1\, \|x'\| \sum_1 Q_{nj}(\rho_n k_n)(\psi(\rho_n j) + \psi(\rho_n k_n))\end{aligned}$$

$$\leqslant 2\beta_1 \|x'\| \psi(\rho_n) e^{-k_n} \sum_1 \frac{k_n^j}{j!}$$

$$\leqslant 2\beta_1 \|x'\| \psi(\rho_n) \cdot \frac{1}{\varepsilon^2 k_n}$$

$$= 2\beta_1 \|x'\| \sqrt[p]{\rho_n}\, \psi(\rho_n) \frac{1}{\varepsilon^2 \sqrt[q]{k_n}\, \sqrt[p]{\rho_n k_n}},$$

where $(1/p) + (1/q) = 1$, and since

$$\sqrt[p]{\rho_n}\, \psi(\rho_n) \leqslant \left(\sum_{j=1}^{\infty} \rho_n \psi^p(j\rho_n) \right)^{1/p} \leqslant \left(\int_0^{\infty} \psi^p(x)\, dx \right)^{1/p}$$

for all $n \in N$, the right side in the last sequence of inequalities (for $\sum_1$) tends to 0 as $n \to \infty$. For $\sum_2$, we have

$$\sum_2 Q_{nj}(\rho_n k_n) \|(F_n^j - F_n^{k_n}) x_n'\|_n \leqslant \rho_n \|A_n x_n'\|_n \sum_2 Q_{nj}(\rho_n k_n) \sum_m \|F_n^m\|_n$$

$$\leqslant \rho_n \|A_n x_n'\|_n \sum_2 Q_{nj}(\rho_n k_n) \sum_m \psi(m\rho_n) = \mathscr{I},$$

where the inner sum is taken over all m satisfying,

$$\min\{j, k_n\} \leqslant m \leqslant \max\{j-1, k_n - 1\}, \qquad 0 \neq j \neq k_n.$$

Now, for $j \in \sum_2$, we have that $(1-\varepsilon) k_n \leqslant j \leqslant (1+\varepsilon) k_n$, and since $t/2 \leqslant \rho_n k_n$ for n large, we see that $(1-\varepsilon)\, t/2 \leqslant (1-\varepsilon)\, \rho_n k_n \leqslant \rho_n j$. So that, in fact, $(1-\varepsilon)\, t/2 \leqslant \rho_n \min\{j, k_n\}$. It follows from this and the assumption that ψ is non-increasing that

$$\sum_m \psi(\rho_n m) \leqslant \psi((1-\varepsilon)\, t/2)\, |j - k_n|.$$

Therefore, applying Schwartz inequality and Lemma 2, we get

$$\mathscr{I} \leqslant \rho_n \|A_n x_n'\|_n \psi((1-\varepsilon)\, t/2) \sum_2 Q_{nj}(\rho_n k_n)\, |j - k_n|$$

$$\leqslant \rho_n \|A_n x_n'\|_n \psi((1-\varepsilon)\, t/2) \left(\sum_{j=0}^{\infty} Q_{nj}(\rho_n k_n) \right)^{1/2}$$

$$\times \left(\sum_{j=0}^{\infty} Q_{nj}(\rho_n k_n)(j - k_n)^2 \right)^{1/2}$$

$$\leqslant \rho_n \sqrt{k_n}\, \|A_n x_n'\|_n \psi((1-\varepsilon)\, t/2),$$

which goes to 0 as $n \to \infty$. Thus (4) is established.

Finally, let $x \in X$ and let $x_n \in X_n$, $(n \in N)$ be such that $\widetilde{\lim}\, x_n = x$. Given $\varepsilon > 0$, there is (by (iii)) an $x' \in D^c$ such that $\|x - x'\| < \varepsilon$. Moreover, there is a sequence (x_n') for which (4) is satisfied and such that $\widetilde{\lim}\, x_n' = x'$. Hence,

$$\begin{aligned} \|P_n T(t)\, x - F_n^{k_n} x_n\|_n \leqslant{} & \|P_n(T(t) - T(\rho_n k_n))\, x\|_n + \|P_n T(\rho_n k_n)(x - x')\|_n \\ & + \|P_n T(\rho_n k_n)\, x' - T_n(\rho_n k_n)\, x_n'\|_n + \|F_n^{k_n}(x_n - x_n')\|_n \\ & + \|(T_n(\rho_n k_n) - F_n^{k_n})\, x_n'\|_n. \end{aligned}$$

The first term on the right side goes to 0 by the strong continuity of $T(t)$ for $t > 0$. Also, $(\rho_n k_n)$ is contained in some interval $[a, b] \subset (0, \infty)$, and since $\|T(t)\|$ is bounded for all $t \geqslant a > 0$, (cf. [8]) the second term is bounded by a constant multiple of ε. The third term tends to 0 as $n \to \infty$ because of the uniform convergence on compacts guaranteed by (3). The fourth term is bounded by $\psi(t/2)\, \|x_n - x_n'\|$ for all large values of n, while the last term tends to 0 in view of (4). Thus (2) is verified. ∎

THEOREM 2. *Let $0 \leqslant \psi$ satisfy the following: For some $\gamma \geqslant 0$, $e^{-\gamma t}\psi(t)$ is non-increasing over $(0, \infty)$, and belongs to $L^1(0, \infty) \cap L^p(0, \infty)$, for some $p > 1$. Further, let (F_n), $F_n \in L(X_n)$, $n = 1, 2, \ldots$ be a sequence of operators and (ρ_n) a null sequence of positive numbers, such that conditions* (i), (ii) *of Theorem* 1 *and* (iii)′ *are satisfied, where* (iii)′ *is the same as* (iii) *but with $\lambda_0 > \omega + \gamma$. Then, $\hat{A}$ is the* i.g. *of a* (1, A) *semigroup $T(t)$, $t > 0$ on X, and* (2) *holds.*

Proof. Put $\tilde{\psi}(t) = e^{-\gamma t}\psi(t)$, $E_n = e^{-\gamma\rho_n} F_n$, $B_n = \rho_n^{-1}(E_n - I)$, and $\hat{B} = \liminf B_n$. The following relations are easily checked:

$$B_n = e^{-\gamma\rho_n} A_n + \gamma_n I, \qquad \text{where} \quad \gamma_n = \rho_n^{-1}(e^{-\gamma\rho_n} - 1) \qquad \text{and} \qquad \hat{B} = \hat{A} - \gamma I.$$

We show that if F_n, A_n, $\hat{A}$, and ψ satisfy the conditions of Theorem 2, then the operators E_n, B_n, $\hat{B}$, and the function $\tilde{\psi}$ satisfy the conditions of Theorem 1 (note that $\tilde{\psi}$ already fulfills the requirements of Theorem 1). (i) holds since

$$\|E_n^k\|_n = \|e^{-\gamma\rho_n k} F_n^k\|_n \leqslant e^{-\gamma\rho_n k}\psi(k\rho_n) = \tilde{\psi}(k\rho_n).$$

For (ii), we observe that

$$\begin{aligned} R(\lambda; B_n) &= \int_0^\infty e^{-\lambda t} e^{tB_n}\, dt = \int_0^\infty e^{-(\lambda - \gamma_n)t} \exp(te^{-\gamma\rho_n} A_n)\, dt \\ &= e^{\gamma\rho_n} \int_0^\infty e^{-\delta_n s} e^{sA_n}\, ds = e^{\gamma\rho_n} R(\delta_n; A_n), \end{aligned}$$

where $s = te^{-\gamma\rho_n}$ and $\delta_n = (\lambda - \gamma_n)\, e^{\gamma\rho_n}$. Hence,

$$\|\lambda R(\lambda; B_n)\|_n = \frac{\lambda}{\lambda - \gamma_n} \|\delta_n R(\delta_n; A_n)\|_n \leqslant C, \qquad \forall \lambda \geqslant \omega, \qquad n \in N.$$

Note that $\delta_n > \lambda \geqslant \omega$, for all n. Finally, it is clear that

$$D^\circ = \{x \in X\colon \exists (x_n), \widetilde{\lim}\, x_n = x, \sup_n \|A_n x_n\|_n < \infty\}$$
$$= \{x \in X\colon \exists (x_n), \widetilde{\lim}\, x_n = x, \sup_n \|B_n x_n\|_n < \infty\},$$

and that $R(\lambda_0 I - \hat{A}) = R((\lambda_0 - \gamma) I - \hat{B})$. Hence (iii) is also satisfied. Therefore, by Theorem 1, there is a (1, A) semigroup $S(t)$, $t > 0$ on X, with i.g. $\hat{B}$ and such that

$$\widetilde{\lim}\, x_n = x \Rightarrow \widetilde{\lim}\, E_n^{[t/\rho_n]} x_n = S(t)\, x, \qquad t > 0.$$

The proof is completed by showing that, in fact,

$$\widetilde{\lim}\, x_n = x \Rightarrow \widetilde{\lim}\, F_n^{[t/\rho_n]} x_n = e^{\gamma t} S(t)\, x, \qquad t > 0.$$

Note that the semigroup $T(t) := e^{\gamma t} S(t)$ is of class (1, A), with i.g. $\hat{A}$ (cf. [8, Theorem 12.2.3]). So suppose that $\widetilde{\lim}\, x_n = x$, then we observe that

$$\begin{aligned} \|F_n^{[t/\rho_n]} x_n - P_n T(t)\, x\|_n &= \|e^{\gamma\rho_n[t/\rho_n]} E_n^{[t/\rho_n]} x_n - e^{\gamma t} P_n S(t)\, x\|_n \\ &\leqslant |e^{\gamma\rho_n[t/\rho_n]} - e^{\gamma t}|\; \|E_n^{[t/\rho_n]}\|_n\, \|x_n\|_n \\ &\quad + e^{\gamma t}\, \|E_n^{[t/\rho_n]} x_n - P_n S(t)\, x\|_n \\ &\leqslant \beta_2\, \|x\|\, \tilde{\psi}(\rho_n[t/\rho_n])\, |e^{\gamma\rho_n[t/\rho_n]} - e^{\gamma t}| \\ &\quad + e^{\gamma t}\, \|E_n^{[t/\rho_n]} x_n - P_n S(t)\, x\|_n. \end{aligned}$$

It is readily seen that the right side tends to 0 as $n \to \infty$ (note that $\tilde{\psi}(\rho_n[t/\rho_n] \leqslant \tilde{\psi}(t/2)$ for n large enough). ∎

COROLLARY 1. *Suppose that* ψ *possesses the properties required by Theorem* 2, *and that hypotheses* (i), (ii), *and* (iii)″ *are fullfilled, where*

(iii)″ *core* A (*cf.* [5]) *and* $R(\lambda_0 I - A)$ *are dense in* X *for some* $\lambda_0 > \omega + \gamma$, *where* $A = \lim A_n$.

Then the conclusion of Theorem 2 *remains valid.*

Proof. Core $A \subset D(A) \subset D(\hat{A}) \subset D^\circ$ and $R(\lambda_0 I - A) \subset R(\lambda_0 I - \hat{A})$. Hence, condition (iii)′ of Theorem 2 satisfied. ∎

COROLLARY 2. (A strong Form of Trotter's Theorem). *Assume the following*:

(a) *The stability condition* (1) *is satisfied,*

(b) D° *and* $R(\lambda_0 I - \hat{A})$ *are both dense in* X *for some* $\lambda_0 > 2\omega$.

Then $\hat{A}$ *is the* i.g. *of a* C_0 *semigroup* $T(t)$, $t \geqslant 0$ *and such that* (2) *holds.*

Proof. Choosing $\psi(t) = Me^{\omega t}$, and γ a number such that $\omega < \gamma < \lambda_0 - \omega$. Then with $\tilde{\psi}(t) = e^{-\gamma t}\psi(t)$, assumptions (i) and (iii)′ of Theorem 2 are satisfied. Also, (1) implies (ii), since for sufficiently large λ

$$\|\lambda R(\lambda; A_n)\|_n \leqslant \lambda \int_0^\infty e^{-\lambda t} \|T_n(t)\|_n \, dt$$

$$\leqslant M\lambda \int_0^\infty e^{(\delta_n - \lambda) t} \, dt$$

$$= \frac{M\lambda}{\lambda - \delta_n}, \qquad \text{where} \quad \delta_n = \frac{e^{\omega \rho_n} - 1}{\rho_n} > 0,$$

and $\delta_n \to \omega$ as $n \to \infty$. Thus, according to Theorem 2, there is a semigroup $T(t)$ satisfying (2) and whose i.g. is $\hat{A}$. Furthermore, from (1) it follows that

$$\|F_n^{[t/\rho_n]}\|_n \leqslant Me^{\omega \rho_n [t/\rho_n]} \leqslant Me^{\omega t}.$$

Hence, by (2), $\|T(t)\| \leqslant M'e^{\omega t}$, $t \geqslant 0$ for some constant M', which shows that $T(t)$ is a C_0 semigroup. ∎

Our last result deals with the case where $e^{-\gamma t}\psi(t)$ is not monotonic for any choice of $\gamma \geqslant 0$. In this case, the result is established for a special choice of the sequence (ρ_n), namely, $\rho_n = 2^{-n}$. We have:

THEOREM 3. *Let* $0 \leqslant \psi$ *and* $\gamma \geqslant 0$ *be such that* $e^{-\gamma t}\psi(t) \in L^1(0, \infty) \cap L^p(0, \infty)$ *for some* $p > 3$. *Further, let* (F_n), $F_n \in L(X_n)$ *be a sequence of operators and* $\rho_n = 2^{-n}$, $n \in N$, *such that* (i)′ *and conditions* (ii), (iii)′ *of Theorem* 2 *are satisfied, where*

(i)′ *There exists an* $x > 0$ *satisfying Corollary* 2, *of Theorem* B, *such that*

$$\|F_n^{[2^n t]}\|_n \leqslant \psi(x + t), \qquad t > 0.$$

Then the conclusion of Theorem 2 *remains valid.*

Proof. As in the proof of Theorem 2, it suffices to consider here the case where $\gamma = 0$. Throughout this proof $\rho_n = 2^{-n}$ so that by Corollary 2 (Chernoff) we have for all $n \in N$,

$$\begin{aligned} &\sum_{j=0}^{\infty} \rho_n \psi(x + \rho_n j) \leqslant L_1 \\ &\sum_{j=0}^{\infty} \rho_n \psi^p(x + \rho_n j) \leqslant L_2, \end{aligned} \tag{5}$$

where L_1, L_2 are some constants.

We note first of all that given positive integers k, n. If we put $t = k\rho_n$, then condition (i)′ implies

$$\|F_n^k\|_n = \|F_n^{[t/\rho_n]}\|_n \leqslant \psi(x+t) = \psi(x + k\rho_n).$$

The proof in this case is parallel to that of Theorem 1. Thus relation (3) is verified in exactly the same way, but here we make use of (5) instead of Lemma 1. To verify (4), we proceed as before: Letting x', (x'_n), and k_n have the same meaning as in the proof of Eq. (4), we get

$$\|\exp(\rho_n k_n A_n) - F_n^{k_n}) x'_n\|_n \leqslant I_1 + I_2.$$

The proof that $I_2 \to 0$ is the same as in Theorem 1. To estimate I_1, we make use of Hölder's inequality and (i)′ above, to find that

$$\begin{aligned} I_1 &\leqslant \rho_n \|A_n x'_n\|_n \sum_{j=1}^{\infty} Q_{nj}(\rho_n k_n) \sum_m \psi(x + m\rho_n) \\ &\leqslant \rho_n \|A_n x'_n\|_n \sum_{j=1}^{\infty} Q_{nj}(\rho_n k_n) \cdot \max_m \psi(x + m\rho_n) \cdot |j - k_n| \\ &\leqslant \rho_n \|A_n x'_n\|_n \left(\sum_{j=1}^{\infty} Q_{nj}(\rho_n k_n) \cdot \max_m \psi(x + m\rho_n) \cdot |j - k_n|^4 \right)^{1/4} \\ &\quad \times \left(\sum_{j=1}^{\infty} Q_{nj}(\rho_n k_n) \cdot \max_m \psi(x + m\rho_n) \right)^{3/4} \\ &= \rho_n \|A_n x'_n\|_n \cdot J_1 \cdot J_2, \end{aligned}$$

where m is as before (see the proof of Theorem 1). Now, since $\rho_n \max_m \psi(x + m\rho_n) \leqslant \sum_{j=0}^{\infty} \rho_n \psi(x + \rho_n j) < L_1$, we find by Lemma 2 that

$$\begin{aligned} J_1 &\leqslant \rho_n^{-1/4} L_1^{1/4} \cdot \left(\sum_{j=1}^{\infty} Q_{nj}(\rho_n k_n)(j - k_n)^4 \right)^{1/4} \\ &= L_1^{1/4} \rho_n^{-3/4} (3\rho_n^2 k_n^2 + \rho_n^2 k_n)^{1/4} \\ &\leqslant L_1^{1/4} \rho_n^{-3/4} (3t^2 + \rho_n t)^{1/4}, \end{aligned}$$

while for J_2 we apply Hölder's equality once more, with $p > 3$ and q the conjugate of p to find that

$$
\begin{aligned}
J_2 &\leqslant \left(\sum_{j=1}^{\infty} Q_{nj}(\rho_n k_n) \right)^{3/4q} \left(\sum_{j=1}^{\infty} Q_{nj}(\rho_n k_n) \cdot (\max_m \psi(x + m\rho_n))^p \right)^{3/4p} \\
&\leqslant \rho_n^{\;3/4p} \left(\sum_{j=1}^{\infty} Q_{nj}(\rho_n k_n) \cdot \max_m \rho_n \psi^p(x + m\rho_n) \right)^{3/4p} \\
&\leqslant \rho_n^{-3/4p} L_2^{3/4p}.
\end{aligned}
$$

Therefore,

$$
I_1 \leqslant \rho_n \, \|A_n P_n x\|_n \cdot J_1 \cdot J_2 \leqslant C_2 C_3 \rho_n^{(p-3)/4p} (3t^2 + \rho_n t)^{1/4},
$$

which goes to 0 as $n \to \infty$, provided that $p > 3$. Thus (4) is established in the present case. The rest of the proof proceeds almost similarly to that of Theorem 1. ∎

Remark. It is possible to replace (i)′ in Theorem 3 by (i), (as noted above the former implies the latter). However, this requires an extra assumption on the function ψ, e.g., if one requires that ψ be bounded on compact subsets of $(0, \infty)$. The only place where this assumption would be used is in evaluating the term $\|F_n^{k_n}(x_n - x_n')\|_n$, which occurs in the last paragraph of the proof (see Theorem 1).

EXAMPLE. Consider the Cauchy problem

$$
\frac{\partial \mathbf{u}}{\partial t} = \mathbf{P}(D)\,\mathbf{u}, \qquad \mathbf{u}(x, 0) = \mathbf{f}(x), \tag{6}
$$

in the Banach space $\mathbf{L}^2(\mathfrak{R}) = L^2(\mathfrak{R}) \times L^2(\mathfrak{R})$, with the standard norm, where $\mathbf{u}(x, t) = (u_1(x, t), u_2(x, t))$ is a vector valued function of the real variables $x \in \mathfrak{R}$, $t > 0$ and $\mathbf{f}(x) = (f_1(x), f_2(x))$, $f_i(x) \in L^2(\mathfrak{R})$ is the given initial condition, and $\mathbf{P}(D)$ is the partial differential operator with respect to x, given by

$$
\mathbf{P}(D) = \begin{pmatrix} D_x^2 + iD_x^4 & iD_x^3 \\ 0 & D_x^2 + iD_x^4 \end{pmatrix}.
$$

The method of discrete approximation corresponding to this problem is described as follows: Let $h > 0$ denote the mesh spacing, and $\rho > 0$ the time increment, then $\partial u/\partial t$ is approximated by the forward difference quotient $\rho^{-1}[u(x, t + \rho) - u(x, t)]$, while the partial derivative $\partial u/\partial x$ with respect to x is approximated by the difference quotient

$$
\Delta_h u(x, t) = (2h)^{-1} \left[u(x + h, t) - u(x, t) \right].
$$

So, the discrete problem corresponding to (6) is

$$\mathbf{u}(x, t+\rho) = \mathbf{F}(\rho)\,\mathbf{u}(x, t), \qquad \mathbf{u}(x, 0) = \mathbf{f}(x), \tag{7}$$

where

$$\mathbf{F}(\rho) = \begin{pmatrix} 1 + \rho\,\Delta_h^2 + i\rho\,\Delta_h^4 & i\rho\,\Delta_h^3 \\ 0 & 1 + \rho\,\Delta_h^2 + i\rho\,\Delta_h^4 \end{pmatrix}.$$

Using the Fourier transform, the discrete problem takes the form

$$\hat{\mathbf{u}}(\xi, t+\rho) = \hat{\mathbf{F}}(\rho)\,\hat{\mathbf{u}}(\xi, t), \qquad \hat{\mathbf{u}}(\xi, 0) = \bar{\mathbf{f}}(\xi),$$

where the Fourier transform of the scalar valued function $u(x, t)$ with respect to x is given as usual by

$$\hat{u}(\xi, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathfrak{R}} e^{-ix\xi} u(x, t)\, dx.$$

The transformed matrix of $\mathbf{F}(\rho)$ is

$$\hat{\mathbf{F}}(\rho) = \begin{pmatrix} 1 - \rho(h^{-1}\sin\xi h)^2 + i\rho(h^{-1}\sin\xi h)^4 & \rho(h^{-1}\sin\xi h)^3 \\ 0 & 1 - \rho(h^{-1}\sin\xi h)^2 + i\rho(h^{-1}\sin\xi h)^4 \end{pmatrix}.$$

Letting (ρ_n), (h_n) be null sequences of positive numbers, we write $\mathbf{F}_n$ for $\mathbf{F}(\rho_n)$ when ρ and h are replaced by ρ_n and h_n, respectively, in the expression defining $\mathbf{F}(\rho)$. Similarly, we write

$$\hat{\mathbf{F}}_n^{[t/\rho_n]} = \begin{pmatrix} (1 + \rho_n\alpha_n)^{[t/\rho_n]} & \rho_n[t/\rho_n](h_n^{-1}\sin\xi h_n)^3\,(1 + \rho_n\alpha_n)^{[t/\rho_n]-1} \\ 0 & (1 + \rho_n\alpha_n)^{[t/\rho_n]} \end{pmatrix},$$

where $\alpha_n = -(h_n^{-1}\sin\xi h_n)^2 + i(h_n^{-1}\sin\xi h_n)^4$.

Using the properties of the multiplier operators in $L^2(\mathfrak{R})$ (see, e.g., [4, VIII, Sect. 3.3]) we find that

$$\|\mathbf{F}_n^{[t/\rho_n]}\| \leqslant \alpha\{a_{11} + a_{12}\},$$

where α is some constant and

$$\begin{aligned} a_{11} &= \sup_\xi |(1 + \rho_n\alpha_n)^{[t/\rho_n]}| \leqslant \kappa \sup_\xi |\exp[t(-\xi^2 + i\xi^4)]| \leqslant \kappa_1, \\ a_{12} &= \sup_\xi |\rho_n[t/\rho_n](h_n^{-1}\sin\xi h_n)^3\,(1 + \rho_n\alpha_n)^{[t/\rho_n]-1}| \\ &\leqslant \kappa_2 \sup_\xi |t\xi^3 \exp[t(-\xi^2 + i\xi^4)]| \leqslant \kappa_3/\sqrt{t}, \end{aligned}$$

where κ_1, κ_2, and κ_3 are constants. The inequalities for a_{11} and a_{12} are verified via standard arguments of calculus. For example, we note that $|t\xi^3 \exp[t(-\xi^2+i\xi^4)]|$ attains its maximum at the point $t\xi^2=3/2$. One should also note when applying Theorem 2 (or its Corollary 1) to the present situation, that $X_n=X=\mathbf{L}^2(\mathfrak{R})$ and $P_n=I$ for all $n\in N$. In this case the notion of limit reduces to the ordinary one. Therefore, setting $\psi(t)=\alpha(\kappa_3/\sqrt{t}+\kappa_1)$ and choosing, e.g., $\gamma=1$, we see that $e^{-t}\psi(t)\in L^1(0,\infty)\cap L^{3/2}(0,\infty)$, and it is decreasing. Thus, $\|\mathbf{F}_n^{[t/\rho_n]}\|_n\leqslant\psi(t)$, which in turn implies condition (i) of Theorem 2 (see also the proof of Theorem 3 and the remark following it). Condition (ii) follows by an argument similar to the one used in proving Eq. (A-5) of [15]. In order to verify (iii)′ of Theorem 2, we use the set $D_1=\{\mathbf{u}\in\mathbf{L}^2(\mathfrak{R}):\hat{\mathbf{u}}$ has compact support$\}$ as a core for $\mathbf{P}(D)$. Note that with $\mathbf{A}_n=\rho_n^{-1}(\mathbf{F}_n-I)$, we have that $\lim \mathbf{A}_n=\mathbf{P}(D)$ in the sense that $\mathbf{A}_n\mathbf{u}\to\mathbf{P}(D)\,\mathbf{u}$, $\mathbf{u}\in D$ in the $\mathbf{L}^2$ norm. This may be verified as in the example of [6, p. 549], by considering the transformed problem where it is shown, e.g., that $\hat{\mathbf{A}}_n\hat{\mathbf{u}}\to P(\xi)\,\hat{\mathbf{u}}$ as $n\to\infty$ on the support of $\hat{\mathbf{u}}$ (for other related details, see [4, p. 532]). It is also known that (iii) is satisfied for all large λ_0 in the case of the operator $\mathbf{P}(D)$. Therefore, by virtue of Corollary 1 of Theorem 2, one concludes the existence of a semigroup $T(t)$, $t>0$ of class $(1, A)$ on $\mathbf{L}^2(\mathfrak{R})$ which solves the Cauchy problem (6), and furthermore, that the solution of the discrete system, namely $\mathbf{F}_n^{[t/\rho_n]}\mathbf{u}(x,0)$, converges to the solution $T(t)\,\mathbf{u}(x,0)$ of (6) as $n\to\infty$.

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